Knowl Inf Syst (2014) 38:259–275
DOI 10.1007/s10115-012-0586-6

REGULAR PAPER

A weighted voting framework for classifiers ensembles

Ludmila I. Kuncheva · Juan J. Rodríguez

Received: 15 August 2011 / Revised: 27 July 2012 / Accepted: 24 November 2012 /
Published online: 7 December 2012

Abstract We propose a probabilistic framework for classifier combination, which gives rigorous optimality conditions (minimum classification error) for four combination methods: majority vote, weighted majority vote, recall combiner and the naive Bayes combiner. The framework is based on two assumptions: class-conditional independence of the classifier outputs and an assumption about the individual accuracies. The four combiners are derived subsequently from one another, by progressively relaxing and then eliminating the second assumption. In parallel, the number of the trainable parameters increases from one combiner to the next. Simulation studies reveal that if the parameter estimates are accurate and the first assumption is satisfied, the order of preference of the combiners is: naive Bayes, recall, weighted majority and majority. By inducing label noise, we expose a caveat coming from the stability-plasticity dilemma. Experimental results with 73 benchmark data sets reveal that there is no definitive best combiner among the four candidates, giving a slight preference to naive Bayes. This combiner was better for problems with a large number of fairly balanced classes while weighted majority vote was better for problems with a small number of unbalanced classes.

Keywords Classifier ensembles · Combination rules · Weighted majority vote · Recall · Naive Bayes

1 Introduction

Classifier ensembles are justly receiving increasing attention and accolade and generating a wealth of research [1,22,23,25,29]. Theoretical and empirical studies have demonstrated

L. I. Kuncheva (✉)
School of Computer Science, Bangor University, Dean Street, Bangor Gwynedd LL57 1UT, UK
e-mail: l.i.kuncheva@bangor.ac.uk

J. J. Rodríguez
Departamento de Ingeniería Civil, Universidad de Burgos, Burgos, Spain
e-mail: jjrodriguez@ubu.es

that an ensemble of classifiers is typically more accurate than a single classifier. Research on classifier ensembles permeate many strands machine learning including streaming data [9,24], concept drift and incremental learning [5].

One of the basic design questions is what combination rule (combiner) to use. Majority vote and weighted majority vote are the most widespread choices when the individual classifiers give label outputs [15]. One of the great assets of the majority vote combiner is that it does not require any parameter tuning once the individual classifiers have been trained. It has been about a decade since Bob Duin posed the question "To Train or not to Train?" [2], and exposed potential caveats in tuning the parameters of the combiner. Choosing the right combiner for the classification problem is not discussed very often, and preferences are given to uncomplicated combiners such as majority vote, average and their weighted versions. Theoretical analyses [7,12,13,16,17,19,28] and experimental comparisons [3,8, 14,27,30,31] of classifier combiners do not offer a definitive answer or a recipe to guide this choice.

Here, we propose a common probabilistic framework for the following four combination methods: majority vote[1] (MV), weighted majority vote (WMV), recall (REC) and naive Bayes (NB). Each combiner is obtained from the previous one when a certain assumption is relaxed or dropped. The price to pay is that each combiner needs more tunable parameters than the previous one. We compare the four combiners on simulated data and on 73 benchmark data sets with a view to propose a strategy to choose among the four combiners.

The rest of the paper is organised as follows. Section 2 introduces the proposed framework and details the four combiners as special cases thereof. Section 3 contains a simulation study and Sect. 4, the experimental protocol and results.

2 A weighted voting framework for classifier ensembles

2.1 Probabilistic set-up

Consider a set of classes $\Omega = \{\omega_1, \ldots, \omega_c\}$ and a classifier ensemble of L classifiers. Denote by s_i the class label proposed by classifier i ($s_i \in \Omega$). We are interested in the probability

$$P(\omega_k \text{ is the true class} \mid s_1, s_2, \ldots, s_L), \qquad k = 1, \ldots, c,$$

denoted for short $P(\omega_k|\mathbf{s})$, where $\mathbf{s} = [s_1, s_2, \ldots, s_L]^T$ is a label vector. Assume that the classifiers give their decisions independently *conditioned upon the class label*,[2] which leads to the following decomposition

$$P(\omega_k|\mathbf{s}) = \frac{P(\omega_k)}{P(\mathbf{s})} \prod_{i=1}^{L} P(s_i|\omega_k) \tag{1}$$

[1] This should be called rather the *plurality* vote because the assigned label is the most voted one, in spite of the fact that majority of more than 50 % may not be reached.

[2] Conditional independence means that

$$P(s_1, s_2, \ldots, s_L|\omega_k) = P(s_1|\omega_k)P(s_2|\omega_k)\ldots P(s_L|\omega_k).$$

However, this assumption precludes unconditional independence, that is,

$$P(s_1, s_2, \ldots, s_L) \neq P(s_1)P(s_2)\ldots P(s_L).$$

Split the product into two parts depending on which classifiers suggested ω_k. Denote by I_+^k the set of indices of classifiers which suggested ω_k, and by I_-^k the set of indices of classifiers which suggested another class label. The probability of interest becomes

$$P(\omega_k|\mathbf{s}) = \frac{P(\omega_k)}{P(\mathbf{s})} \times \prod_{i \in I_+^k} P(s_i = \omega_k|\omega_k) \times \prod_{i \in I_-^k} P(s_i = \overline{\omega}_k|\omega_k) \quad (2)$$

All four combiners described next rely upon the same conditional independence assumption. They differ on the following assumption about the individual accuracies of the classifiers. If the assumption is met, the respective combiner is *optimal* in the sense that it guarantees the minimum Bayes error.

- **Equal individual accuracies.** When $P(s_i = \omega_k|\omega_k) = p$ and $P(s_i = \omega_j|\omega_k) = \frac{1-p}{c-1}$, for any $i = 1, \ldots, L$, $k, j = 1, \ldots, c$ $j \neq k$, then majority vote is the optimal combination rule. Note that for the optimality to hold, not only the accuracies should be equal but also the "leftover" should be uniformly distributed across the remaining classes.
- **Different individual accuracies.** When $P(s_i = \omega_k|\omega_k) = p_i$ and $P(s_i = \omega_j|\omega_k) = \frac{1-p_i}{c-1}$, for any $k, j = 1, \ldots, c$ $j \neq k$, then the weighted majority vote is the optimal combiner with weights as derived in Sect. 2.3.
- **Different individual class-specific recalls.** When $P(s_i = \omega_k|\omega_k) = p_{ik}$ and $P(s_i = \omega_j|\omega_k) = \frac{1-p_{ik}}{c-1}$, for any $k, j = 1, \ldots, c$ $j \neq k$, then the recall combiner is the optimal combiner. The details are derived in Sect. 2.4.
- **Different confusion matrices.** When $P(s_i = \omega_j|\omega_k) = p_{ijk}$, then the Naive Bayes combiner is the optimal combiner.

The optimality of the combiner, however, is asymptotic, and holds for sample size approaching infinity. For finite sample sizes, the accuracy of the estimates of the parameters may be the primary concern. A combiner with fewer tunable parameters may be preferable even though its optimality assumption does not hold. This issue will appear later as an important lesson from the experimental study.

2.2 Majority vote (MV)

To give a correct label, "proper" majority vote requires that more than 50 % of the voters give the correct label. If all classifiers have the same accuracy $P(s_i = \omega_k|\omega_k) = p$ for any $i = 1, \ldots, L$ and $k = 1, \ldots, c$, then the majority vote will be correct if $L_{\text{maj}} = \lfloor \frac{L}{2} \rfloor + 1$ or more votes are correct. Then,

$$P_{\text{proper MV}} = \sum_{i=L_{\text{maj}}}^{L} \binom{L}{i} p^i (1-p)^{(L-i)} \quad (3)$$

The Condorcet Jury Theorem, dated back in 1785 (cited after [26]), states that

1. If $p > 0.5$, then $P_{\text{proper MV}}$ is monotonically increasing and tends to 1 as $L \to \infty$.
2. If $p < 0.5$, then $P_{\text{proper MV}}$ is monotonically decreasing and tends to 0 as $L \to \infty$.
3. If $p = 0.5$, then $P_{\text{proper MV}} = 0.5$ for any L.

Lam and Suen [18] analyse the cases of odd and even L and the effect on the ensemble accuracy of adding or removing classifiers. Shapley and Grofman [26] note that the result is valid even for unequal individual accuracies, provided their distributions are symmetrical about the mean. Matan [21] gives tight upper and lower bounds of the majority vote accuracy in the case of unequal individual accuracies.

Here, we consider majority vote in the wider sense of the word, as a synonym of plurality vote. In this case, there is no requirement that more than 50 % of the voters are correct for the majority to be correct. If there are many classes, a much smaller percentage may suffice. In the absence of further information about the classifiers, assume that all incorrect labels "share" the misclassification probability, that is,

$$P(s_i = \omega_j|\omega_k)_{j \neq k} = \frac{(1-p)}{(c-1)}. \tag{4}$$

Substituting in the probabilistic framework defined in (2),

$$P(\omega_k|\mathbf{s}) = \frac{P(\omega_k)}{P(\mathbf{s})} \times \prod_{i \in I_+^k} p \times \prod_{i \in I_-^k} \frac{1-p}{c-1} \tag{5}$$

$$= \frac{P(\omega_k)}{P(\mathbf{s})} \times \prod_{i \in I_+^k} p \times \prod_{i \in I_-^k} \frac{1-p}{c-1} \times \frac{\prod_{i \in I_+^k} \frac{1-p}{c-1}}{\prod_{i \in I_+^k} \frac{1-p}{c-1}} \tag{6}$$

$$= \frac{P(\omega_k)}{P(\mathbf{s})} \times \prod_{i \in I_+^k} \frac{p(c-1)}{1-p} \times \prod_{i=1}^{L} \frac{1-p}{c-1} \tag{7}$$

Notice that $P(\mathbf{s})$ and the last product term in (7) do not depend on the class label. The prior probability, $P(\omega_k)$, does depend on the class label but not on the votes, so it can be designated as the *class constant*. Rearranging and taking the logarithm,

$$\log(P(\omega_k|\mathbf{s})) = \log\left(\frac{(1-p)^L}{P(\mathbf{s})(c-1)^L}\right) + \log\left(P(\omega_k)\right) + \log\left(\frac{p(c-1)}{1-p}\right) \times |I_+^k|, \tag{8}$$

where $|.|$ denotes cardinality. Dividing by $\log\left(\frac{p(c-1)}{1-p}\right)$ and dropping all terms that do not depend on the class label or the vote counts, (8) becomes

$$\log(P(\omega_k|\mathbf{s})) \propto \underbrace{\log\left(\frac{1-p}{p(c-1)}\right)\log\left(P(\omega_k)\right)}_{\text{class constant } \zeta(\omega_k)} + |I_+^k|. \tag{9}$$

Note that $|I_+^k|$ is the number of votes for ω_k. Choosing the class label corresponding to the largest posterior probability is equivalent to choosing the class most voted for, subject to a constant term. Interestingly, the standard majority vote rule does not include a class constant, and is still one of the most robust and accurate combiners for classifier ensembles. Besides, including the class constant will make MV a trainable combiner, which eliminates one of its main assets. Since one of our aims is to give practical recommendations, in the experiments in this study, we adopted the standard majority vote formulation, whereby the class label is obtained by

$$\omega = \arg\max_k |I_+^k|. \tag{10}$$

2.3 Weighted majority vote (WMV)

The weighted majority vote is among the most intuitive and widely used combiners [11,20]. It is the designated combination method derived from minimising a bound on the training error in AdaBoost [4,6]. Freund and Schapire [6] offer a similar probabilistic explanation as an alternative justification for the weights in the two-class version of AdaBoost. Here, we use our framework to derive the multi-class version of the weighted majority vote, and specify the conditions for its optimality.

The weighted majority vote follows from relaxing the assumption about equal individual accuracies. Thus, it will be the optimal combiner when the accuracies are equal as well, and the MV combiner is its exact reduced version. Let $P(s_i = \omega_k|\omega_k) = p_i$ and $P(s_i = \omega_j|\omega_k) = \frac{1-p_i}{c-1}$, for any $k, j = 1, \ldots, c$ $j \neq k$. Following the same derivation path as with MV, Eq. (2) becomes

$$P(\omega_k|\mathbf{s}) = \frac{P(\omega_k)}{P(\mathbf{s})} \times \prod_{i \in I_+^k} p_i \times \prod_{i \in I_-^k} \frac{1-p_i}{c-1} \tag{11}$$

$$= \frac{P(\omega_k)}{P(\mathbf{s})} \times \prod_{i \in I_+^k} \frac{p_i(c-1)}{1-p_i} \times \prod_{i=1}^{L} \frac{1-p_i}{c-1} \tag{12}$$

$$= \frac{1}{P(\mathbf{s})} \times \prod_{i=1}^{L} \frac{1-p_i}{c-1} \times P(\omega_k) \times \prod_{i \in I_+^k} \frac{p_i(c-1)}{1-p_i}. \tag{13}$$

Then,

$$\log(P(\omega_k|\mathbf{s})) = \log\left(\frac{\prod_{i=1}^{L}(1-p_i)}{P(\mathbf{s})(c-1)^L}\right) + \log\left(P(\omega_k)\right) + \sum_{i \in |I_+^k|} \log\left(\frac{p_i}{1-p_i}\right) + |I_+^k| \times \log(c-1). \tag{14}$$

Dropping the first term, which will not influence the class decision, and expressing the classifier weights as

$$w_i = \log\left(\frac{p_i}{1-p_i}\right), \quad 0 < p_i < 1,$$

Equation (14) leads to

$$\log(P(\omega_k|\mathbf{s})) \propto \underbrace{\log\left(P(\omega_k)\right)}_{\text{class constant } \zeta(\omega_k)} + \sum_{i \in |I_+^k|} w_i + |I_+^k| \times \log(c-1). \tag{15}$$

If $p_i = p$ for all $i = 1, \ldots, L$, Eq. (15) reduces to the majority vote Eq. (8).

2.4 Recall combiner (REC)

The next logical step in relaxing the assumptions is to allow different probabilities of correct classification depending on the classifier *and* the class, $P(s_i = \omega_k|\omega_k) = p_{ik}$. This amounts to different individual class-specific *recalls*. The idea is that each class is considered separately versus the union of the remaining classes. We assume again that the misclassification probability is shared among the remaining (wrong) classes, that is, $P(s_i = \omega_j|\omega_k) = \frac{1-p_{ik}}{c-1}$, for any $k, j = 1, \ldots, c$, $j \neq k$. Starting again with Eq. (2),

$$P(\omega_k|\mathbf{s}) = \frac{P(\omega_k)}{P(\mathbf{s})} \times \prod_{i \in I_+^k} p_{ik} \times \prod_{i \in I_-^k} \frac{1-p_{ik}}{c-1} \tag{16}$$

$$= \frac{P(\omega_k)}{P(\mathbf{s})} \times \prod_{i \in I_+^k} \frac{p_{ik}(c-1)}{1-p_{ik}} \times \prod_{i=1}^{L} \frac{1-p_{ik}}{c-1} \tag{17}$$

This time the last product depends on the class label ω_k but not on the decisions in $\mathbf{s}$. Therefore, it will be part of the class constant. Rearranging and taking the logarithm,

$$\log(P(\omega_k|\mathbf{s})) = \log\left(\frac{1}{P(\mathbf{s})(c-1)^L}\right) + \log(P(\omega_k)) + \sum_{i=1}^{L} \log(1-p_{ik})$$
$$+ \sum_{i \in |I_+^k|} \log\left(\frac{p_{ik}}{1-p_{ik}}\right) + |I_+^k| \times \log(c-1). \tag{18}$$

Dropping the first term, and denoting the recall weights by

$$v_{ik} = \log\left(\frac{p_{ik}}{1-p_{ik}}\right), \quad 0 < p_{ik} < 1,$$

we arrive at

$$\log(P(\omega_k|\mathbf{s})) \propto \underbrace{\log(P(\omega_k)) + \sum_{i=1}^{L} \log(1-p_{ik})}_{\text{class constant } \zeta(\omega_k)} + \sum_{i \in |I_+^k|} v_{ik}$$
$$+ |I_+^k| \times \log(c-1). \tag{19}$$

If $p_{ik} = p_i$ for any $k = 1, \ldots, c$, Eq. (19) reduces to the weighted majority vote Eq. (15).

To best of our knowledge, the recall combiner has not been used before. It arose from the logical sequence of relaxing the assumption of equal individual accuracies, falling between two well- known combiners: the weighted majority vote and naive Bayes.

2.5 Naive Bayes combiner (NB)

The Naive Bayes combiner has been acclaimed for its rigorous statistical underpinning and robustness. We can derive this combiner by finally dropping the assumption of equal individual accuracies, that is, allowing for $P(s_i = \omega_j|\omega_k) = p_{ijk}$. We can think of p_{ijk} as the (j, k)th entry in a probabilistic confusion matrix for classifier i in the ensemble. In this case,

$$\log(P(\omega_k|\mathbf{s})) = \log\left(\frac{1}{P(\mathbf{s})}\right) + \log(P(\omega_k)) + \sum_{i=1}^{L} \log(P(s_i|\omega_k)) \tag{20}$$

$$\propto \underbrace{\log(P(\omega_k))}_{\text{class constant } \zeta(\omega_k)} + \sum_{i=1}^{L} \log(p_{i,s_i,k}). \tag{21}$$

Table 1 Scopes of optimality (denoted by a black square) and the number of tunable parameters of the 4 combiners for a problem with c classes and an ensemble of L classifiers

Combiner	1	2	3	4	Number of parameters
Majority vote	■	–	–	–	none
Weighted majority vote	■	■	–	–	$L + c$
Recall	■	■	■	–	$L * (c + 1)$
Naive Bayes	■	■	■	■	$L * c^2 + c$

Column headings: *1* Equal p, *2* Classifier-specific p_i, *3* Classifier- and class-specific p_i, *4* Full confusion matrix

2.6 Overview of the four combiners

The progressive relaxation of the assumption means that the combiners have a nested optimality scope. The enlargement of the optimality scope is paid by acquiring more tunable parameters. Table 1 shows the optimality scopes and the number of tunable parameters for each combiner. The additional c parameters are for estimating the prior probabilities for the classes.

It is tempting to use always Naive Bayes because it has the largest optimality scope. In practice, however, the success of a particular combiner will depend partly on the assumptions and partly on the availability of sufficient data to make reliable estimates of the parameters. Non-optimal but more robust combiners may fare better than the optimal combiner. The simulation and the experimental studies described next highlight the importance of this issue when choosing a combiner.

Curiously, the well-known and widely used majority vote, weighted majority vote and Naive Bayes combiners typically ignore the class constant (Eqs. (9), (15), and (21), respectively). This means that these combination methods will be optimal only if we add to their current set of assumptions the assumption that the classes are equiprobable. The same argument holds for the recall combiner Eq. (19), but this combiner came as a byproduct of the proposed framework and does not enjoy the popularity of the other three combiners.

3 A simulation study

3.1 Protocol

Experiments with simulated classifier outputs were carried out as follows:

- Number of classes $c \in \{2, 3, 4, 5, 10, 20, 50\}$;
- Number of classifiers $L \in \{2, 3, 4, 5, 10, 20, 50\}$;
- Number of instances (labels) 500;
- Number of runs 100.

For each run, c classes were generated by labelling the 500 instances according to a symmetric Dirichlet distribution.[3] To enforce class-conditional independence, the classifiers in the ensemble were constructed class by class as shown in the Algorithm in Fig. 1. To form the label set of classifier i and class k, take the labels for class k and replace a percentage between 0 and 66.7 % with labels randomly sampled from Ω. The c sets of labels for each

[3] Each set of c random numbers summing up to 1 had the same chance of being generated.

```
Inputs:   number of instances N, number of classes c, number of classifiers L

% 'True' class labels (tl - vector with N elements)
    rp ← random permutation of the integers from 1 to N
    si ← [1, sort(first c − 1 values of rp), N] % breakpoints of the classes
    for i = 1,...,c;
       set the values of tl from si(i) to si(i+1) equal to i;
    endfor % class labels

% Ensemble class labels (el - matrix with N rows and L columns)
    for j = 1,...,L
       cl ← tl % copy the true labels in cl
       pr ← random proportion of indices to perturb between 0 and 2/3
       for i = 1,...,c
          identify the elements of cl equal to i
          replace a random set = proportion pr of these elements
          with random integers between 1 and c
       endfor
       store cl as the j-th column of matrix el
    endfor
```

Fig. 1 Algorithm for generating 'true' class labels and class-conditional independent outputs of the base classifiers for one experimental run

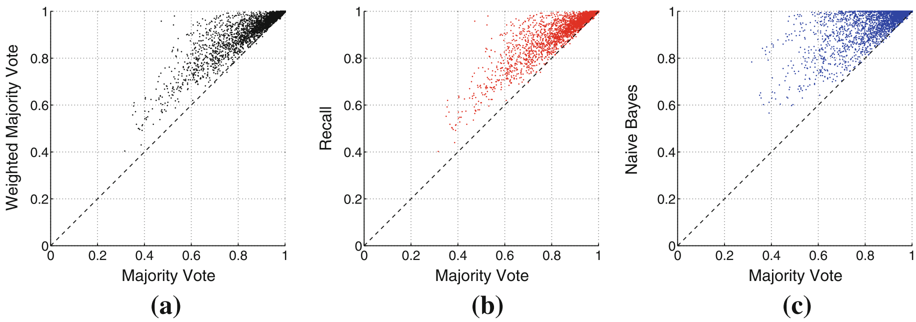


Fig. 2 Relationship between the ensemble accuracies using the majority vote as the benchmark combiner. Each scatterplot contains 4900 ensembles points

classifier are concatenated to form the final output of classifier i. L such classifiers were generated and the four combination rules were calculated. The classification accuracies for each pair (L, c) were averaged across the 100 runs.

3.2 Results

Figure 2 shows the relationship between the combiners' accuracies for the whole ranges of L and c. Since there are 100 runs and 7 values of each parameter, there are 4900 ensemble accuracies for each combiner. The figure shows that the weighted majority and the recall combiner are similar, with the recall combiner having an edge over the WMV. They are both better than the majority vote combiners and worse than Naive Bayes combiner.

Figure 3 gives the ensemble accuracies as a function of $\log(c)$ for 2 and for 50 classifiers. The benefit from the NB combiner is clearly demonstrated by the upward trend of the case

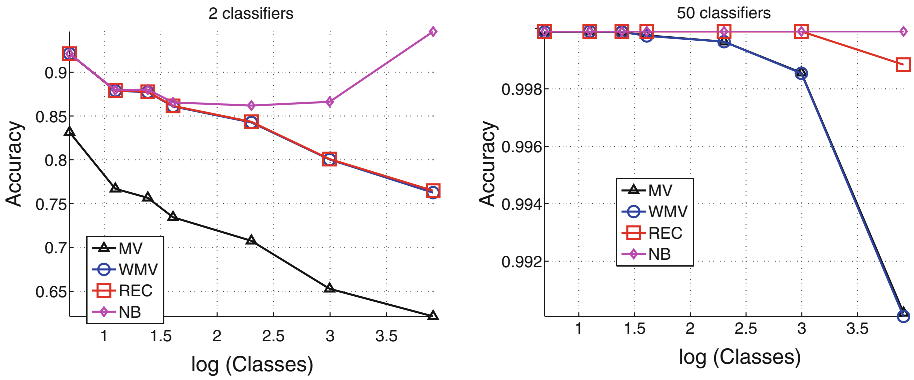


Fig. 3 Ensemble accuracies of the 4 combiners as a function of log(c) (exact parameter estimates)

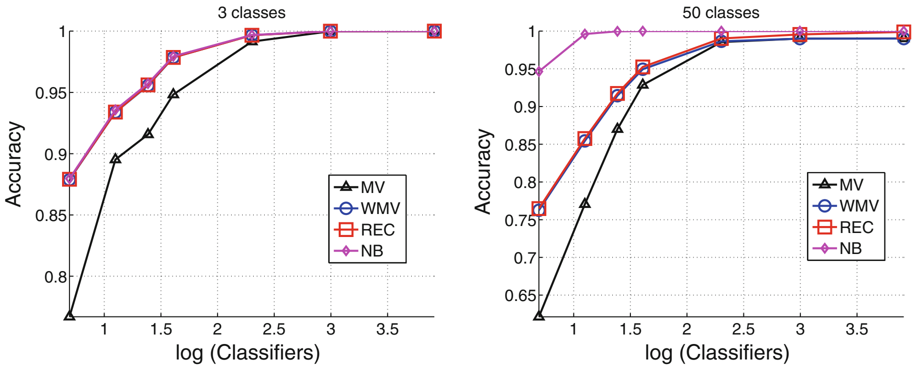


Fig. 4 Ensemble accuracies of the 4 combiners as a function of log(L) (exact parameter estimates)

of two classifiers and increasing number of classes. The other three combiners show inferior result. The same tendency is valid to a lesser extent for $L = 50$ classifiers.

The dependency of the combiner accuracies on the number of classifiers is shown in Fig. 4. For 2 classes, REC and NB are identical, hence we chose to show 3 and 50 classes. The figure suggests that for a small number of classes, NB is almost identical to REC and WMV. For a large number of classes, however, NB is clearly the winner of the four combiners.

The order of the four combiners in all simulations is as expected (from best to worst): NB, REC, WMV and MV. The NB combiner seems to have a great advantage over the other three combiners for larger number of classes. However, NB may suffer from the curse of dimensionality. In this part of the simulation study, we assumed that there is no noise in the estimates of the parameters of the combiners. Any noise could be very harmful to NB's accuracy. REC and WMV will be less vulnerable, and MV will be immune to the size of the validation data used for estimating the parameters. To demonstrate this effect, we ran simulations with 10 % random noise on the labels *after* the estimates of the parameters have been computed. Figures 5 and 6 show the same graphs as in Figs. 3 and 4 for the noisy estimates.

For a small L and c, NB is still the best combiner but its accuracy plummets for larger L and c. One reason for this drastic change of behaviour is that the number of examples here was limited to 100. In real data, the number of instances is usually much larger, and the

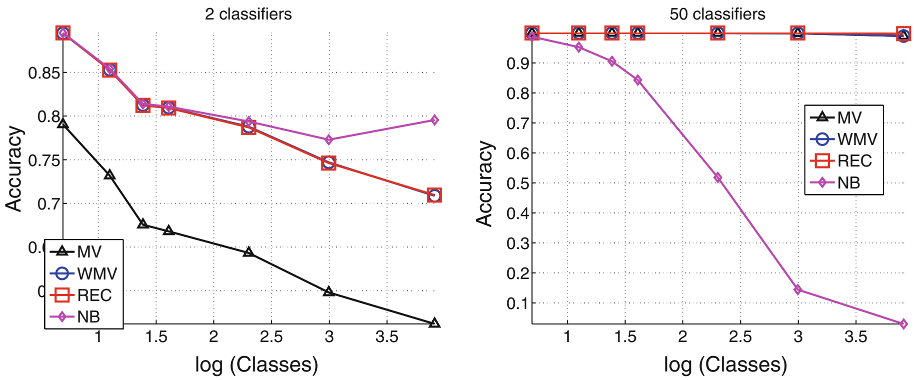


Fig. 5 Ensemble accuracies of the 4 combiners as a function of $\log(c)$ (10 % label noise)

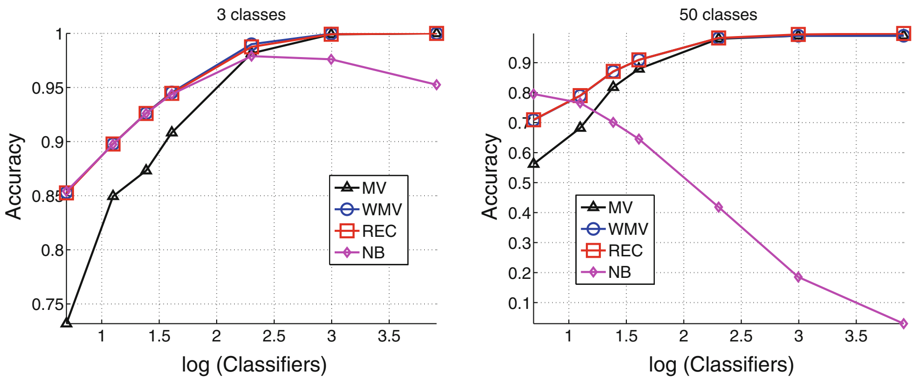


Fig. 6 Ensemble accuracies of the 4 combiners as a function of $\log(L)$ (10 % label noise)

stability-plasticity balance is much more delicate. If there is sufficient unseen data to allow a "good" estimate of the parameters of the NB combiner, then NB should be better than the remaining three combiners. However, in real data sets, the assumption of class-conditional independence of the classifier outputs may not be true, which will affect the preference of the combiners in an unforeseen way.

4 Experiments with real data

4.1 Protocol

In real data sets, the independence assumption is likely to be violated but the type of dependencies that may occur in practice is too difficult to incorporate into a simulation study. The purpose of this study was to define the optimality framework giving rise to the four combiners. While they behaved as expected in the simulation study, the combiners may be differently affected by violation of the assumptions. The purpose of the experiments was to find out whether there is a dominant combiner across a wide variety of pattern classification problems, and to recommend a choice depending on some characteristic of the data set.

The four combiners were tested on 73 publicly available benchmark data sets as indicated in Table 2. We used $L = 100$ decision tree classifiers. To guard against $\log(0)$ and division by 0, we set a rounding threshold $t = 10^{-8}$ for all classification accuracies, as well as for the estimates of the prior probabilities. All estimates which were less than t were reassigned to t, and all estimates greater than $1 - t$ were reset to $1 - t$. For each scenario, we carried out 10 replicas of 10-fold cross-validation. For each cross-validation fold, the training set was split into two equal parts called "proper" training and validation. All individual classifiers were trained on bootstrap samples from the proper training part and all parameters of the combiners were evaluated on the validation part except for the prior probabilities which were estimated from the whole training part of the fold.

All experiments were run within the Weka environment [10]. The accuracy of each ensemble is the average across the 100 testing results.

4.2 Results

Table 3 shows the ensemble accuracies. The best accuracies for each data set are underlined. We have indicated the winner even where, due to rounding, the values for the data set appear as identical in the table. The bottom row of the table shows the average accuracies across the data sets. With the diversity of domains and types of data sets in this experiment, and the large span of classification accuracies, it is unlikely that these accuracies will be commensurable. But even though the average values across the data sets cannot serve as a valid performance gauge, they give a rough reference of the achievements of the combiners.

The table shows that there is no clear winner, hence we calculated the ranks for the combiners. For example, on the dermatology data set, NB receives rank 1 (the best), REC receives rank 2, WMV receives rank 3 and MV rank 4 (the worst). In case of a tie, the ranks are shared. The average ranks across the data sets were: MV 2.5205, WMV 2.4315, REC 2.8356 and NB <u>2.2123</u>, making NB the best combiner. The Friedman nonparametric ANOVA was run on the ranks, followed by a multiple comparisons test. The p value of Friedman's ANOVA was 0.0269 indicating significant differences among the ranks. It was subsequently found that NB is significantly better than REC. One possible reason for the poor performance of REC is hinted by the simulation results. The results with REC are similar to these with WMV but the number of tunable parameters of REC is approximately c times larger. With inaccurate estimates of the parameters, the small performance advantage of REC over WMV may be smeared. In addition, it is not clear how the violation of the assumption of conditional independence affects the performance of the combiners. Then, it is not surprising the REC combiner has not surfaced thus far.

Figure 7 shows the distribution of the "winner" between NB and WMV, the combiner with the second best rank. The axes are the prior of the largest class and the prior of the smallest class. The feasible space is within a triangle, as shown in the figure. The right edge corresponds to the 2-class problems because the smallest and the largest prior sum up to 1. The number of classes increases from this edge towards the origin (0, 0). The left edge of the triangle corresponds to equiprobable classes. The largest prior on this edge is equal to the smallest prior, which means that all classes have the same prior probabilities. This edge can be thought of as the edge of balanced problems. The balance disappears towards the bottom right corner. The pinnacle of the triangle corresponds to two equiprobable classes.

Each data set is represented as a dot. The marker denotes the better combiner: circles markers for NB and triangle markers for WMV. The figure does not shows a clear pattern of dominance of one combiner over the other. NB seems to be slightly better for larger number of classes and for generally balanced classes while WMV is better for problems with fewer unbalanced classes. If we applied the NB combiner for $c > 3$ and WMV for $c \leq 3$ for the 73 data sets, it will give average accuracy of 83.33 % and will achieve the top rank.

Table 2 Data sets from UCI used in this study

Data set	#	#E	#N	#D	#C	R
Abalone	1	4,177	7	1	28	0.002
Anneal	2	898	6	32	5	0.012
Arrhythmia	3	452	206	73	13	0.008
Audiology	4	226	0	69	24	0.018
Autos	5	205	15	10	6	0.045
Balance	6	625	4	0	3	0.170
Breast-w	7	699	9	0	2	0.526
Breast-y	8	286	0	9	2	0.423
Bupa	9	345	6	0	2	0.725
Car	10	1,728	0	6	4	0.054
Cmc	11	1,473	2	7	3	0.529
Credit-a	12	690	6	9	2	0.802
Credit-g	13	1,000	7	13	2	0.429
Crx	14	690	6	9	2	0.802
Dermat	15	366	1	33	6	0.179
Dna	16	3,186	0	180	3	0.463
Ecoli	17	336	7	0	8	0.014
Glass	18	214	9	0	6	0.118
Heart-c	19	303	6	7	2	0.836
Heart-h	20	294	6	7	2	0.564
Heart-s	21	123	5	8	2	0.070
Heart-stat	22	270	13	0	2	0.800
Heart-v	23	200	5	8	2	0.342
Hepatitis	24	155	6	13	2	0.260
Horse-colic	25	368	7	15	2	0.586
Hypo	26	3,163	7	18	2	0.050
Hypothyroid	27	3,772	6	21	4	0.001
Ionosphere	28	351	34	0	2	0.560
Iris	29	150	4	0	3	1.000
Kr-vs-kp	30	3,196	0	36	2	0.915
Krk	31	28,056	6	0	18	0.006
Labour	32	57	8	8	2	0.541
Led-24	33	5,000	0	24	10	0.925
Letter	34	20,000	16	0	26	0.903
Lrs	35	531	93	0	10	0.004
Lympho	36	148	3	15	4	0.025
Mfeat-fact	37	2,000	216	0	10	1.000
Mfeat-fourier	38	2,000	76	0	10	1.000
Mfeat-karh	39	2,000	64	0	10	1.000
Mfeat-morph	40	2,000	6	0	10	1.000
Mfeat-pixel	41	2,000	0	240	10	1.000
Mfeat-zernike	42	2,000	47	0	10	1.000
Mushroom	43	8,124	0	22	2	0.931
Nursery	44	12,960	0	8	5	0.001

Table 2 continued

Data set	#	#E	#N	#D	#C	R
Optdigits	45	5,620	64	0	10	0.969
Page	46	5,473	10	0	5	0.006
Pendigits	47	10,992	16	0	10	0.922
Phoneme	48	5,404	5	0	2	0.415
Pima	49	768	8	0	2	0.536
Primary	50	339	0	17	21	0.012
Promoters	51	106	0	57	2	1.000
Ringnorm	52	300	20	0	2	0.841
Sat	53	6,435	36	0	6	0.408
Segment	54	2,310	19	0	7	1.000
Shuttle	55	58,000	9	0	7	0.000
Sick	56	3,772	7	22	2	0.065
Sonar	57	208	60	0	2	0.874
Soy-small	58	47	0	35	4	0.588
Soybean	59	683	0	35	19	0.087
Splice	60	3,190	0	60	3	0.463
Tae	61	151	3	2	3	0.942
Threenorm	62	300	20	0	2	1.000
Tic-tac-toe	63	958	0	9	2	0.530
Twonorm	64	300	20	0	2	0.974
Vehicle	65	846	18	0	4	0.913
Vote1	66	435	0	15	2	0.629
Voting	67	435	0	16	2	0.629
Vowel-context	68	990	10	2	11	1.000
Vowel-nocntxt	69	990	10	0	11	1.000
Waveform	70	5,000	40	0	3	0.977
Yeast	71	1,484	8	0	10	0.011
Zip	72	9,298	256	0	10	0.456
Zoo	73	101	1	15	7	0.098

#E: examples, #N: numeric attributes, #D: discrete attributes, #C: classes, R: ratio between the smallest and largest class priors

Table 3 Ensemble accuracies with the 4 combiners

Data	MV	WMV	REC	NB
1	24.1	24.1	24.5	24.7
2	97.9	97.9	98	98.6
3	73.1	73.2	71.6	69.8
4	74.6	74.4	73.8	75.0
5	69.9	70.2	70.7	76.3
6	82.9	82.9	82.2	82
7	95.8	95.8	96.0	96.0
8	71.5	71.7	70	70
9	68.6	68.8	67.2	67.2
10	88.8	88.8	88	90.0
11	52.4	52.5	52	52.1

Table 3 continued

Data	MV	WMV	REC	NB
12	85.8	85.8	85.8	85.8
13	73.8	73.9	70.9	70.9
14	85.9	86	86.0	86.0
15	94.8	95	95.5	96.3
16	94.1	94.1	94	94
17	82.3	82.2	81.3	82.9
18	71	71	70.5	73.0
19	79.9	79.5	79.2	79.2
20	79.7	79.7	80.1	80.1
21	93.2	93.3	84.3	84.3
22	80.7	80.7	80.5	80.5
23	73.8	74.0	69.3	69.3
24	81.5	81	79.7	79.7
25	84.7	84.8	84.3	84.3
26	99.1	99.1	99.2	99.2
27	99.5	99.5	99.2	99.5
28	91.6	91.5	91.7	91.7
29	93.4	93.3	93.5	93.7
30	99	99.1	99.1	99.1
31	74.9	74.9	74.9	75.3
32	81.5	82.4	84.5	84.5
33	73.9	73.9	74.1	74.3
34	90.9	90.9	90.7	91.3
35	85.3	85.3	83.2	82.9
36	77.3	78.2	76.6	76.6
37	93.2	93.2	93	93.3
38	79.4	79.4	79	79.1
39	89.5	89.4	89.4	89.1
40	72.3	72.3	72.5	73.0
41	80.5	80.5	80.6	81.2
42	75.2	75.1	74.4	78.0
43	100	100	100.0	100.0
44	95.7	95.7	94.7	95.3
45	95	95.0	94.9	95.0
46	97.1	97.1	97	96.8
47	97.5	97.5	97.5	97.6
48	87.1	87.1	86.6	86.6
49	75.7	75.7	75.5	75.5
50	40.4	40.5	39.9	38.3
51	80.8	80.2	79.8	79.8
52	87	86.8	87.1	87.1
53	89.5	89.5	89.1	88.9
54	96.1	96.1	96.1	96.4
55	99.9	99.9	99.9	99.9

Table 3 continued

Data	MV	WMV	REC	NB
56	98.4	98.4	98.2	98.2
57	75.9	75.9	76.2	76.2
58	95.8	95.8	95.8	96.4
59	89.8	89.9	91.6	93.2
60	93.2	93.2	93.3	93.3
61	52.4	53.1	53.9	52.3
62	77.2	76.7	77.1	77.1
63	86	86	86.2	86.2
64	87.7	87.6	87.9	87.9
65	72.4	72.4	71.6	72.5
66	89.4	89.3	89.4	89.4
67	95.6	95.5	95.5	95.5
68	80.4	80.4	80.4	85.3
69	81.2	81.3	81.3	85.2
70	83	83	83.0	83.0
71	59.1	59.1	59	59.1
72	91.9	91.9	91.7	91.8
73	91.4	91.2	91.3	92.8
average	83.01	83.03	82.64	83.04

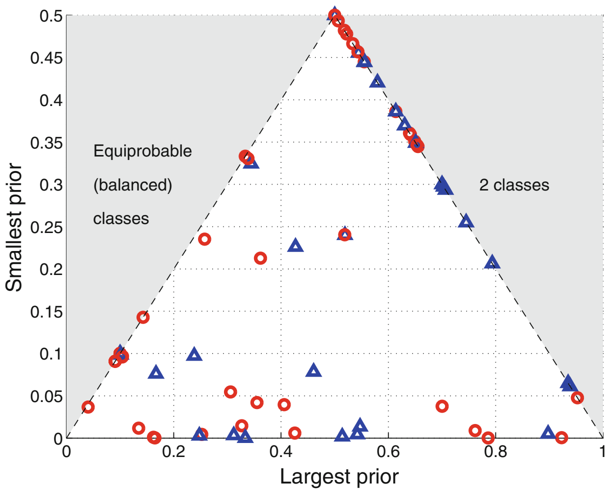


Fig. 7 Winning combiner for the 73 data sets. *Circles* markers denote NB and *triangle* markers denote WMV

5 Conclusions

This paper introduces a formal framework for classifier combination when the classifiers produce label outputs which are conditionally independent. Under this framework, we derive the optimality conditions for four combination methods: majority vote (MV), weighted majority vote (WMV), the recall combiner (REC) and Naive Bayes (NB). Simulations were carried out to examine the combiners and highlight some practical concerns. Experiments with 73 benchmark data sets revealed that there is no dominant combiner. NB was the most successful combiner overall but the differences with MV and WMV were not found to be statistically significant. NB has the widest optimality scope but also the largest number of parameters to train among the four combiners. The simulation study showed that NB can suffer badly from inaccurate estimates of its parameters. The experimental results with real data did not show such anomalies with NB, suggesting that, in practice, the data are usually sufficient for obtaining reasonable parameter estimates.

The differences between the performances of the four combiners in the simulation study were blurred in the experiments with real data for at least two reasons. Beside the noise of parameter estimates, the second reason for this is that the conditional independence assumption may not be satisfied. It would be interesting to study to what extent this assumption holds, and how it affects the combiners. Another direction for future research is how the combiners are influenced by: the choice of base classifier, the ensemble sizes, the method of producing the base classifiers and the ensemble diversity. It is also interesting to explore other characteristics of the data sets and find better niches for the four combiners.

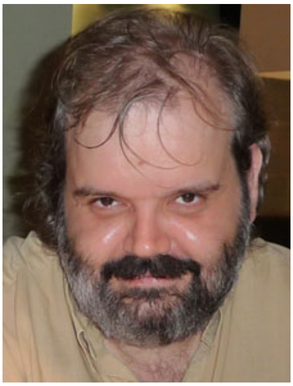
References

1. Brown G (2010) Ensemble learning. In: Sammut C, Webb G (eds) In encyclopedia of machine learning. Springer, Berlin
2. Duin RPW (2002) The combining classifier: to train or not to train? In: Proceedings 16th international conference on pattern recognition, ICPR'02, Canada, pp. 765–770
3. Duin RPW, Tax DMJ (2000) Experiments with classifier combination rules. In: Kittler J, Roli F (eds) Multiple classifier systems, vol. 1857 of lecture notes in computer science. Springer, Italy, pp. 16–29
4. Eibl G, Pfeiffer KP (2005) Multiclass boosting for weak classifiers. J Mach Learn Res 6:189–210
5. Elwell R, Polikar R (2011) Incremental learning of concept drift in nonstationary environments. IEEE Trans Neural Netw 22(10):1517–1531
6. Freund Y, Schapire RE (1996) Experiments with a new boosting algorithm. Thirteenth international conference on machine learning. Morgan Kaufmann, San Francisco, pp. 148–156
7. Fumera G, Roli F (2005) A theoretical and experimental analysis of linear combiners for multiple classifier systems. IEEE Trans Pattern Anal Mach Intell 27:942–956
8. Ghosh K, Ng YS, Srinivasan R (2011) Evaluation of decision fusion strategies for effective collaboration among heterogeneous fault diagnostic methods. Comput Chem Eng 35(2):342–355
9. Grossi V, Turini F (2012) Stream mining: a novel architecture for ensemble-based classification. Knowl Inf Syst 30:247–281
10. Hall M, Frank E, Holmes G, Pfahringer B, Reutemann P, Witten IH (2009) The WEKA data mining software: An update, SIGKDD explorations 11
11. Kim H, Kim H, Moon H, Ahn H (2011) A weight-adjusted voting algorithm for ensembles of classifiers. J Korean Stat Soc 40(4):437–449
12. Kittler J, Hatef M, Duin RPW, Matas J (1998) On combining classifiers. IEEE Trans Pattern Anal Mach Intell 20(3):226–239
13. Kuncheva L (2002) A theoretical study on six classifier fusion strategies. IEEE Trans Pattern Anal Mach Intell 24(2):281–286
14. Kuncheva LI (2003) 'Fuzzy' vs 'non-fuzzy' in combining classifiers designed by boosting. IEEE Trans Fuzzy Syst 11(6):729–741
15. Kuncheva LI (2004) Combining pattern classifiers. Methods and algorithms. Wiley, New York
16. Kuncheva L, Whitaker C, Shipp C, Duin R (2003) Limits on the majority vote accuracy in classifier fusion. Pattern Anal Appl 6:22–31
17. Lam L, Suen C (1995) Optimal combination of pattern classifiers. Pattern Recognit Lett 16:945–954
18. Lam L, Suen C (1997) Application of majority voting to pattern recognition: an analysis of its behavior and performance. IEEE Trans Syst Man Cybern 27(5):553–568
19. Lin X, Yacoub S, Burns J, Simske S (2003) Performance analysis of pattern classifier combination by plurality voting. Pattern Recognit Lett 24(12):1795–1969
20. Lingenfelser F, Wagner J, André E (2011) A systematic discussion of fusion techniques for multi-modal affect recognition tasks. In: Proceedings of the 13th international conference on multimodal interfaces, ICMI '11. ACM, New York, pp. 19–26

21. Matan O (1996) On voting ensembles of classifiers (extended abstract). In: Proceedings of AAAI-96 workshop on integrating multiple learned models, pp. 84–88
22. Polikar R (2006) Ensemble based systems in decision making. IEEE Circuits Syst Mag 6:21–45
23. Re M, Valentini G, (2011) Ensemble methods: a review, Data mining and machine learning for astronomical applications, Chapman & Hall, London (in press)
24. Read J, Bifet A, Holmes G, Pfahringer B (2012) Scalable and efficient multi-label classification for evolving data streams. Mach Learn 88(1–2, SI):243–272
25. Sewell M (2011) Ensemble learning, Technical Report RN/11/02. Department of Computer Science, UCL, London
26. Shapley L, Grofman B (1984) Optimizing group judgemental accuracy in the presence of interdependencies. Public Choice 43:329–343
27. Tax DMJ, Duin RPW, van Breukelen M (1997) Comparison between product and mean classifier combination rules. In: Proceedings workshop on statistical pattern recognition, Prague, Czech Republic
28. Tumer K, Ghosh J (1999) Combining artificial neural nets. In: Sharkey A (ed) Linear and order statistics combiners for pattern classification. Springer, London, pp 127–161
29. Xu L, Krzyzak A, Suen CY (1992) Methods of combining multiple classifiers and their application to handwriting recognition. IEEE Trans Syst Man Cybern 22:418–435
30. Zhang CX, Duin RP (2011) An experimental study of one- and two-level classifier fusion for different sample sizes. Pattern Recognit Lett 32(14):1756–1767
31. Zhang L, Zhou WD (2011) Sparse ensembles using weighted combination methods based on linear programming. Pattern Recognit 44(1):97–106

Author Biographies

Ludmila I. Kuncheva received the MSc degree from the Technical University of Sofia, Bulgaria in 1982, and the Ph.D. degree from the Bulgarian Academy of Sciences in 1987. Until 1997, she worked at the Central Laboratory of Biomedical Engineering at the Bulgarian Academy of Sciences. She is currently a Professor at the School of Computer Science, Bangor University, UK. Her interests include pattern recognition and classification, machine learning and classifier ensembles. She has published two books and above 200 scientific papers.



Juan J. Rodríguez received the BS, MS and Ph.D. degrees in Computer Science from the University of Valladolid, Spain, in 1994, 1998 and 2004, respectively. He worked with the Department of Computer Science, University of Valladolid from 1995 to 2000. Currently, he is working with the Department of Civil Engineering, University of Burgos, Spain, where he is an Associate Professor. His main interests are machine learning, data mining and pattern recognition. He is a member of the IEEE Computer Society.