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A fuzzy consensus aggregation operator

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Abstract

An aggregation operator of quantitative opinions about the acceptance or rejection of a certain alternative is proposed. The main idea is to include a degree of consensus between the experts in computing the final value. We aim at strengthening the acceptance or rejection rate if the experts agree in their assessments. The proposed operator is directly applicable to the two-level classification paradigm where a pool of classifiers is used to infer the decision. The operator allows us to form a complex classification boundary in the space of experts' (classifiers') decisions expressed in terms of degrees of membership. The connective can easily be extended to define a classifier with a "refuse-to-decide" option.

Keywords: Fuzzy decision making; Consensus; Pattern recognition and classification; Aggregation; Multiple classifiers

1. Introduction

The aggregation theme in fuzzy decision making is a broad one [10] comprising a lot of aggregation connectives, both general and specific, single-level and hierarchical [7, 19], covering nearly the whole range from totally pessimistic through totally optimistic ends of the scale. This variety stems from the difference in aims, strategies, and hypotheses with respect to the type of opinions, their interrelations and relevance.

One class of aggregation paradigms is based on the notion of consensus. Consensus (general agreement on an opinion) has figured prominently in decision making since the early ages of development of humanity, determining social and political changes, life imprisonment, vital medical conclusions, etc. It has been pointed out by Day [8] that consensus models are potentially fertile and can be applied in various domains.

Consensus methods concern predominantly voting schemes but their scope has been expanded to preference relations [4, 11–18, 29, 31], Bayesian inference and group decisions [28], rank ordering [30], structures [6, 9], linguistically defined assessments [24], direct estimates [14, 20–23], Boolean phrases [5], etc. Considering the problem of finding a set of acceptable (due to a certain consensus criterion) alternatives in a probabilistic setting (replacing "alternatives" with "events", and "preference" with "certainty") the so called consensus theory can be applied [1–3, 25]. It studies the combination of probabilities obtained from different sources (e.g., experts) into a single probability distribution on the set of events.

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Although consensus is traditionally meant as a full and unanimous agreement we will adhere to a more relaxed view stating that consensus is a measurable parameter whose highest value corresponds to unanimity, and the lowest one – to a complete disagreement. It seems not necessary to bring more arguments to support the need of a continuous-valued degree of consensus: many excellent arguments have been reported in the literature (see, e.g., [18]). The main aim in defining a measure of consensus is twofold. The first aspect is to assess how far the group of experts (or decision makers) is from the unanimous agreement, thus giving a tool to monitor the evolution of group preferences [4, 6, 24, 29, 31]. The second aim is to form a set of alternatives (issues, options, decisions, variants) that obey certain consensus requirements. This set may either be considered as the overall solution or, in turn, searched through. Taking into account the nonstatistical nature of the experts' assessments, the most appropriate mathematical tool to handle calculations of this type appears to be fuzzy set theory. A notion of fuzzy linguistic majority has been used in [15, 16] to derive a fuzzy set of alternatives satisfying statements such as "Most experts are convinced of accepting this alternative". This view has further evolved into a so-called "soft" consensus that may be expressed by the statement "Most of the important experts agree on almost all of the relevant alternatives" [11–13, 17, 18].

In this paper we propose an aggregation connective that contains explicitly the degree of consensus between the decision makers. We define an axiomatic framework based on some heuristic rationale. A two-level pattern recognition paradigm is considered where each first-level decision maker yields a degree of support in the interval [0, 1] for the hypothesis that the object being classified belongs to a given class. By applying the proposed operator we aim at a better classification performance than that of a two-level classifier with the same topology and a classical aggregation connective. A refuse-to-decide option is considered and the classification performance is evaluated in this aspect. The degree of consensus is estimated using some operations on fuzzy sets.

2. Consensus aggregation operator

Let us consider the following statement drawing a parallel between multi-classifier pattern recognition and decision making. Let $\Omega = \{\omega_1, \ldots, \omega_M\}$ be the set of classes (alternatives) and let x be an object generated by one of them. The problem to be solved is to choose the correct class knowing the classification decisions (individual preferences) of n classifiers (decision makers). Let $\boldsymbol{R} = \{R_1, \ldots, R_n\}$ be the set of classifier and let r_{ij} denote the assessment of the ith classifier with respect to the jth class. In the trivial case the first-level classifier is supposed to indicate one class label, i.e., $r_{ij} \in \{0, 1\}$, and

$$\sum_{j=1}^{M} r_{ij} = 1, \quad i = 1, \ldots, n$$

(the orthogonality condition).

We call the matrix $[r_{ij}]$ the *"decision profile"* [21, 23]. The problem of inferring the correct class is literally the one of choosing the best alternative. Depending on what r_{ij} means different techniques should be applied. Constituting a probabilistic framework, one could use the consensus theory mentioned in the introduction. Looking at the values r_{ij} as degrees of acceptance, support, compliance, typicality, severity, strength of confirmation, etc., fuzzy aggregation operators appears to be more appropriate.

In what follows we will process one alternative (classification hypothesis) at a time. We stipulate that we can consider classes separately, i.e., their mutual dependence (not necessarily exclusiveness) can be handled at a higher decision level. In other words, the support for one class will be inferred regardless of the support which the classification scheme renders to other classes. There are a lot of practical problems that are either essentially two-class (e.g., signature verification [26]), or multiclass with not mutually exclusive classes. The latter means that an object may belong with different degrees to more than one class. Many such examples could be drawn from medical diagnosis where numerous diseases may occur simultaneously, and where formulating of mutually exclusive classes is both computationally ineffective and medically not interpretable. The interrelations between such classes are more complex than mutual exclusiveness because one disease may provoke, prevent, complicate, impede, etc. another one. This fact determines our choice to decompose the M-class problem, thus aiming at a refining of the single-class decision before passing it to the higher level analysis.

The problem is to label x as a member (or non-member) of a given class ω. The decision profile reduces to a vector $\boldsymbol{y} = [y_1, \ldots, y_n]^{\mathrm{T}} \in [0,1]^n$ with y_i denoting the degree of support provided by the ith classifier for the hypothesis that x belongs to ω. The greater the value, the stronger the evidence for the class. The ensuing task is to aggregate the outputs in order to obtain a final degree of support. This degree can then be compared with a certain threshold in order to make the final decision.

Let us assume that an aggregation connective has already been chosen depending on the character of the classification decisions. We propose to include explicitly the degree of agreement (called *degree of consensus*) between the decision makers into the computation of the final decision. This idea is based on the following rationale:

- if the decisions agree on an aggregated value above certain threshold T from $(0,1)$ we could increase the strength of support;
- if the decisions agree on an aggregated value below T we could even more "depress" the support;
- if the decisions disagree (regardless if the resultant value is greater or less than T) there are no reasons to change the aggregated value in either direction.

Obviously, the behavior of the consensus aggregation operator is drastically different below and above the threshold. Using a "standard" aggregation operator the agreement between the participants can hardly be captured.

Let us denote by $K(A(\boldsymbol{y}), C(\boldsymbol{y}), T) \in [0,1]$ the consensus aggregation operator with A being the aggregated value and C, the degree of consensus.

It is reasonable for $K(A(\boldsymbol{y}), C(\boldsymbol{y}), T)$ to comply with the following axiomatic properties (see, e.g., [27]):

(*i*) *Symmetry.* Let $\boldsymbol{y}'$ be a vector containing an arbitrary permutation of the components of $\boldsymbol{y}$. We impose symmetry on $K(A(\boldsymbol{y}), C(\boldsymbol{y}), T)$, i.e., $K(A(\boldsymbol{y}'), C(\boldsymbol{y}'), T) = K(A(\boldsymbol{y}), C(\boldsymbol{y}), T)$. If both the aggregation operator and the consensus measure are symmetric, the above is satisfied since K is not a direct function of $\boldsymbol{y}$.

(*ii*) *Selective monotonicity on the degree of consensus.* $K(A(\boldsymbol{y}), C(\boldsymbol{y}), T)$ is a nondecreasing function of $C(\boldsymbol{y})$ provided $A(\boldsymbol{y}) = \text{const} > T$; $K(A(\boldsymbol{y}), C(\boldsymbol{y}), T)$ is a nonincreasing function of $C(\boldsymbol{y})$ provided $A(\boldsymbol{y}) = \text{const} < T$.

(*iii*) *Unanimity of the aggregation.* Let $y_1 = y_2 = \cdots = y_n = t$. Then

(*iii.*1). $A(\boldsymbol{y}) = t$. This stems from the *unanimity property* (idempotency) of the aggregation [27]);

(*iii.*2). $C(\boldsymbol{y}) = 1$; and

(*iii.*3). $K(t, 1, T) = t$.

(*iiia*) *Strengthened Unanimity of the aggregation.* Let $y_1 = y_2 = \cdots = y_n = t$. Then

(*iiia.*1). $A(\boldsymbol{y}) = t$;

(*iiia.*2). $C(\boldsymbol{y}) = 1$; and

(*iiia.*3). $K(t, 1, T) > t$, for $t > T$; $K(t, 1, T) < t$, for $t < T$; $K(T, 1, T) = T$.

Obviously, the properties (iii) and (iiia) can be satisfied only alternatively. The *strengthened unanimity* corresponds to a certain degree to the Archimedean property which underlies strict operations [10].

It deserves mentioning that the above set of axioms differs from the usual setting for an aggregation connective because the participating values $A(\boldsymbol{y})$, $C(\boldsymbol{y})$, and T are meant to play completely different roles in the operator.

The rule may be constructed in different ways. Three formulas are presented below:

$$K(A(\boldsymbol{y}), C(\boldsymbol{y}), T) = T + (A(\boldsymbol{y}) - T)C(\boldsymbol{y}). \tag{1}$$

Calculated in this way $K(A(\boldsymbol{y}), C(\boldsymbol{y}), T)$ belongs to $[0, 1]$ because

$$\begin{aligned} K(A(\boldsymbol{y}), C(\boldsymbol{y}), T) &= T + A(\boldsymbol{y})C(\boldsymbol{y}) - TC(\boldsymbol{y}) \\ &= (1 - C(\boldsymbol{y}))T + A(\boldsymbol{y})C(\boldsymbol{y}) \geqslant 0. \end{aligned}$$

Since $K(A(\boldsymbol{y}), C(\boldsymbol{y}), T)$ is a convex combination of T and $A(\boldsymbol{y})$ (both less than 1) with the coefficients summing up to 1, the following inequality holds:

$$\begin{aligned} 0 &\leqslant \min\{A(\boldsymbol{y}), T\} \leqslant K(A(\boldsymbol{y}), C(\boldsymbol{y}), T) \\ &\leqslant \max\{A(\boldsymbol{y}), T\} \leqslant 1, \quad \forall C(\boldsymbol{y}) \in [0, 1]. \end{aligned}$$

Its symmetry is guaranteed by definition, since K depends explicitly only on A and C and not on their arguments.

K satisfies (ii) (*selective monotonicity*) because

$$\partial K/\partial C(y) = A(y) - T < 0 \quad \text{for } A(y) < T,$$

$$\partial K/\partial C(y) = A(y) - T > 0 \quad \text{for } A(y) > T.$$

Unanimity property is obviously satisfied. A similar formula that satisfies the *strengthened unanimity* property is as follows:

$$S = T + \operatorname{sgn}(A(y) - T)\, f(|A(y) - T|)\, C(y),$$

where $f(\cdot)$ is a monotonic function of its argument. This can be, e.g. $\sqrt{|A(y) - T|}$, or more generally $|A(y) - T|^{1/p}$, with $p > 1$. Then

$$K(A(y), C(y), T) = \begin{cases} 1, & \text{if } S \geqslant 1, \\ 0, & \text{if } S \leqslant 0, \\ S, & \text{otherwise.} \end{cases} \tag{2}$$

Another formula that obeys the same axioms is:

$$K(A(y), C(y), T) = \frac{1}{1 + \dfrac{1 - T}{T}\exp(-\alpha C(y)(A(y) - T))} \tag{3}$$

where α is a scaling constant.

The symmetry of the above operator is obvious. Since the derivative of K on $C(y)$ is positive for $A(y) > T$ and negative otherwise, the axiom for the *selective monotonicity* is satisfied. We introduce this expression since it practically satisfies the *strengthened unanimity* for large values of α, and moreover, it is differentiable.

Fig. 1 shows the behavior of K versus $A(y)$ with four different values of $C(y)$ and with threshold value $T = 0.4$. From the figure it appears that if we introduce certain threshold of acceptance of the alternative (e.g., $T_{\text{accept}} = 0.76$) the decisive parameter is the degree of consensus. For example, using Eq. (1), with $A(y) = 0.67$, the alternative will not be accepted whatever the consensus is. According to Eq. (2) the alternative will be accepted only if the consensus is 0.7 or higher. Similarly, Eq. (3) with $\alpha = 10$ allows for acceptance of the alternative if the degree of support has been achieved with a consensus degree greater than, e.g., 0.6.

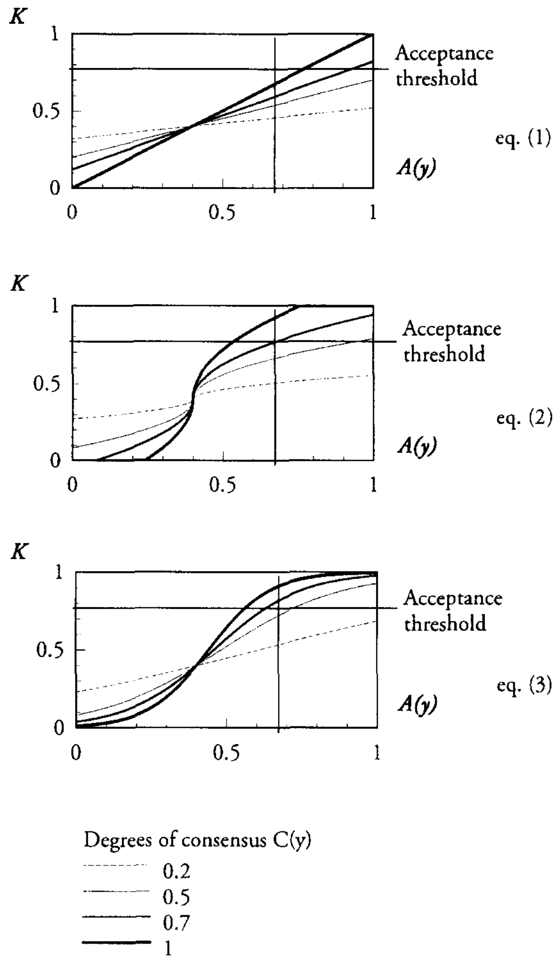

Fig. 1.

This graphical example highlights the ability of the decision operator to be influenced directly by the degree of consensus regardless of the aggregated value $A(y)$. The higher the degree of consensus, the more certain the decision, either positive or negative.

The considerations up to here were focused on the formal expression of the fuzzy consensus aggregation rule. The questions remaining outside the exposition are the one of selection of the aggregation connective $A(y)$, and the one of measuring the degree of consensus $C(y)$ among the n degrees of support of the alternative. Numerous comprehensive studies on fuzzy aggregation can be found, and the selection of a proper connective can be based on some initial information available in the task. In what follows we will use some of the most widely used aggregation operations and some consensus measures from [21] where the problem of assessing the degree of consensus is addressed (without being exhaustively studied).

3. A graphical illustration

In order to clarify the idea we will provide a graphical illustration of the ability of the proposed operator to form more complex, and at the same time intuitively pleasing, decision boundaries in the space of individual assessments when compared with some of the classical aggregation rules.

Let us consider the following settings:

(1) Let $\mathbf{y} = [y_1, y_2]^T \in [0,1]^2$ be the vector containing the decision support for the alternative provided by two first-level decision makers (classifiers). Let $y_i = 0$ denote total rejection and $y_i = 1$, total acceptance, $i = 1, 2$. This will enable us to visualize the results on the 2-D plane.

(2) Since there are many different ways to introduce competence levels of the experts (classifiers) we will confine the discussion to the case of equal competence.

(3) After calculating the degree of support we have to make a crisp decision either accepting or rejecting the alternative. We wish also to supply the rule with the ability to reply "refuse-to-decide" which can be an advantage in the multi-alternative (multiclass) case. This can simply be done by choosing two thresholds: T_{accept} and T_{reject} and implementing the rule:

$$\text{Final Decision} = \begin{cases} \text{accept}, & \text{if FS} > T_{\text{accept}}, \\ \text{reject}, & \text{if FS} < T_{\text{reject}}, \\ \text{refuse to decide}, & \text{otherwise}. \end{cases}$$

The notation FS is used here for the value of the final support. In the case of classical aggregation connectives it will be $A(\mathbf{y})$. In our case FS stands for $K(A(\mathbf{y}), C(\mathbf{y}), T)$.

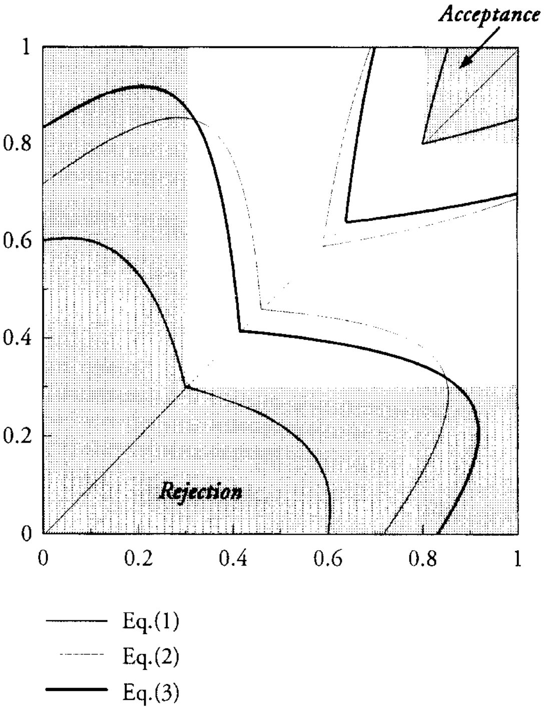

Fig. 2.

(4) As a measure of consensus we will use the *highest discrepancy* [21] which for the current statement is:

$$C(\mathbf{y}) = 1 - |y_1 - y_2|.$$

Figs. 2–4 show the isoquants of $K(A(\mathbf{y}), C(\mathbf{y}), T)$ with the values for the constants $T_{\text{accept}} = 0.8$, $T_{\text{reject}} = 0.3$, and $T = 0.5$. The aggregation rules depicted are as follows:

- Fig. 2. The minimum aggregation rule, $A(\mathbf{y}) = \min(y_1, y_2)$. The areas for acceptance and rejection are highlighted in order to make the comparison of the proposed formulas with the original aggregation rule more visible. The Eqs. (1)–(3) have been used to calculate the respective curves for $K(A(\mathbf{y}), C(\mathbf{y}), T) = T_{\text{accept}}$ and $K(A(\mathbf{y}), C(\mathbf{y}), T) = T_{\text{reject}}$. The square root is used as the function f in (2)
- Fig. 3. The maximum aggregation rule, $A(\mathbf{y}) = \max(y_1, y_2)$.
- Fig. 4. The average aggregation rule, $A(\mathbf{y}) = 0.5(y_+, y_2)$.

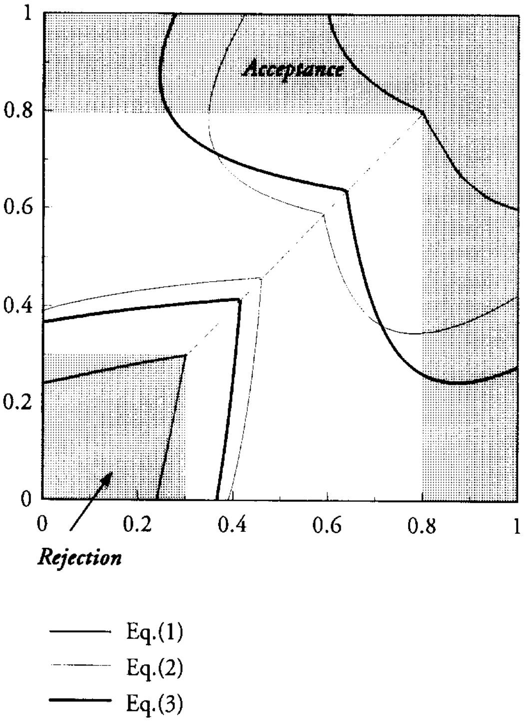

Fig. 3.

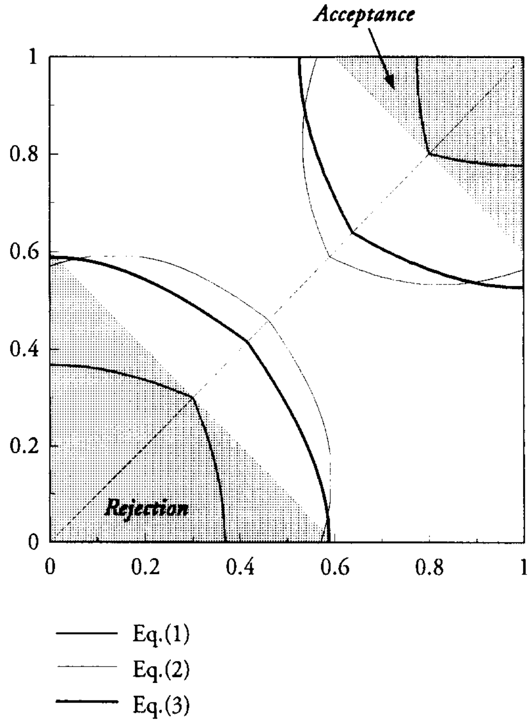

Fig. 4.

The regions for acceptance and rejection, and the isoquants for K calculated with the three equations are shown in all three figures in the same style. Purposefully we have chosen one example from each attitude category [10]: the conjunctive, disjunctive and compromise attitudes.

Fig. 5 shows the decision surface of the average aggregation rule and its "distortion" with the proposed formulas.

What can be seen from the example is that:

1. The fuzzy consensus operator in general provides more complex boundaries of the decision regions for acceptance and rejection that hopefully expresses better the human insight on the problem.

2. It can be observed in all three figures that the first rule (Eq. (1)) defines a subdomain of the initial region. That means that the rule is more "strict", or restrictive. It accepts or rejects the alternative only if the value of the background aggregation rule is appropriate *and* the degree of consensus between the decision makers is acceptable.

3. The two rules defined by the Eqs. (2) and (3) exhibit a more "generous" behavior due to the *strengthened unanimity* property. Thus an alternative may be accepted even if the value of the aggregation rule is below the acceptation threshold but if the decision makers agree on it. On the other hand, taking the consensus into account results into cutting out the farthest parts of the acceptance and rejection regions.

4. As it can be expected, Fig. 3 is the dual of Fig. 2. The noncompensative character of the aggregation rules (Figs. 2 and 3) is preserved in the proposed fuzzy consensus operator.

By introducing certain multiplicative and/or power coefficients expressing different competence of experts we can "skew" the pictures but the character of the behavior will be the same.

4. Experimental example

The proposed rule is hardly comparable with the aggregation rules based on consensus since they

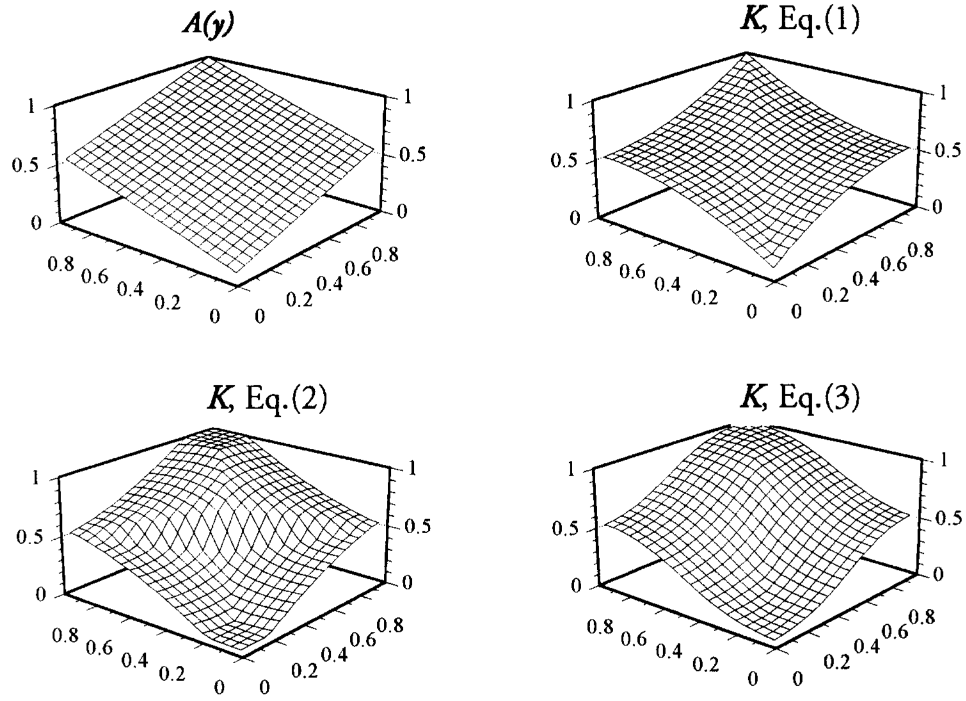

Fig. 5.

operate on preference relations while the considered one uses direct assessments of the support of the classification hypothesis. Furthermore, they do not explicitly use the degree of consensus in calculating the final value. Finally, one can always find data that support one's claim of superiority of the proposed rule to the others. Having this in mind, an illustrative example is presented using a small real data set from neonatal medicine.

The problem of detecting a hyaline membrane disease of a newborn is of vital importance because this decision determines ultimately the further manipulations necessary to save the life of the infant. Normally, the clinic manifestation of the disease is highly obscured, the relevant X-ray examination is informative only after the disease is in its late stages, the great bulk of the easily measurable parameters are quite nonspecific. All this outlines the complexity of the diagnostic task. The experiment presented in this paper has only an illustrative meaning and is not supposed to be medically sound.

There is a high risk a preterm newborn to be affected by this particular disease due to the immaturity of the lungs. The problem is to predict if the newborn will suffer from hyaline membrane disease or not, i.e. we have to distinguish between two-classes: *affected* and *healthy*. The set of features used here include: gestation age, morphological maturity, and the Apgar index at two subsequent time moments after delivery. We will refer to the features as X_1, X_2, X_3, and X_4. The sample consists of 99 cases: 51 healthy and 49 affected, each one described by values of the features and by the respective class label.

Each feature will be considered as an "expert". In order to form the first-level decisions we built a classifier on each individual feature using linear discriminant analysis (LDA). The degree of support of the hypothesis of being affected was measured by the posterior probability as estimated by the LDA program. The output values of these single-feature classifiers are regarded as y_i's: the individual classification decisions subject to aggregation.

The main goal of the experiment was to compare the behavior of the proposed operator with that of the respective aggregation connective embedded in the formula. As in the previous illustration we will assume that the experts (classifiers) are equally competent.

The *integral highest discrepancy* measure of consensus [21] has been used which for this case is

$$C(y) = 1 - \max_{i=1,\ldots,4} |y_i - \underline{y}|$$

where $\underline{y}$ is the mean value of y_i's.

The following aggregation connectives have been tried: minimum, maximum, and average. Each operator has been applied first individually, and then through the proposed fuzzy consensus aggregation operators corresponding to Eqs. (1)–(3), and denoted K_1, K_2, and K_3, respectively. Having fixed the acceptance and rejection thresholds, each aggregation technique has been used to infer the crisp decision for any of the objects. The points for which FS (K or $A(\boldsymbol{y})$) exceeded T_{accept} have been assigned "accept" (corresponding to class *affected*) while those below T_{reject} have been assigned "reject" (corresponding to class *healthy*). The points in-between have been designated for refuse. We counted the apparent classification accuracy P_C for the objects admitted to classification

$$P_C = N_C / N_{\text{not-refused}}$$

where N_C is the number of correctly classified objects (those with a preliminary label *affected* and being assigned "acceptance" by the classifier, and those with label *healthy* and being assigned "reject"), and $N_{\text{not-refused}}$ is the number of all objects adopted for a crisp classification decision.

Since the classification accuracy strongly depends on the threshold values, the experiment consisted in a smooth changing of the acceptance and rejection thresholds in a conjugated manner, i.e. $T_{\text{accept}} = 1 - T_{\text{reject}}$, so that T_{accept} increases from 0.5 to 1, and T_{reject} decreases. In order to assess the classification performance accounting simultaneously for classification accuracy and refuse rate, the following criterion has been used:

$$U = \lambda P_C + (1 - \lambda)(1 - R),$$

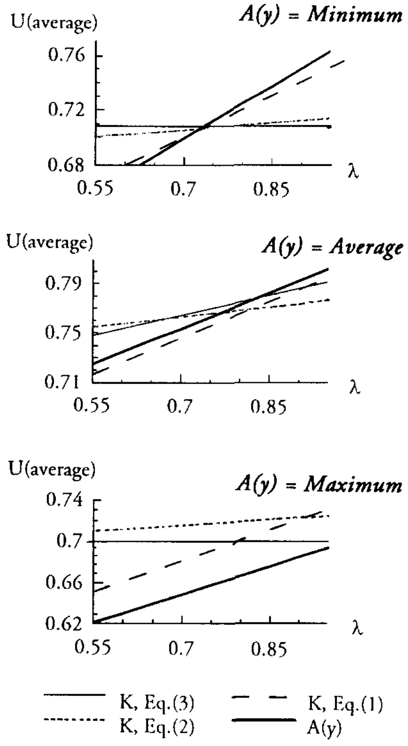

Fig. 6.

where R is an estimate of the overall probability of refuse, and λ is a parameter in [0, 1]. By fixing λ and changing monotonically T_{accept} (respectively T_{reject}) a $P_C(R)$ curve is obtained for each aggregation connective. In order to assess the curve we measured the average values of U for different λ, i.e., with different prevalence of the accuracy to refuse rate. The higher the value of U, the better the two-level classifier, and, consequently – the aggregation connective used. We restricted the refuse rate R to vary up to 0.8 since for its higher values the assessment of the accuracy is based on too few objects and the result may be spurious. Fig. 6 shows the average values of U as a function of λ for the accuracy–refuse curves obtained with the aggregation operators under investigation. It can be seen that for certain values of λ the proposed operator strongly dominates the embedded rule, whichever of the three formulas is used. This is especially persuasive with the maximum aggregation connective. It should be mentioned that the presented example is not a sufficient background to formulate general conclusions but only a piece of evidence in support of the main idea.

5. Conclusions

In this paper a fuzzy consensus aggregation operator has been proposed which aims at providing a more flexible tool to aggregate pointwise expert assessments for a certain alternative. The main rationale underlying the proposed operator is that the degree of consensus should be directly used in the computation of the final support. Three formulas have been proposed in a heuristically defined axiomatic framework. The ability of such an operator to generate complex decision boundaries in the space of individual decisions has been illustrated. The operator seems to yield plausible results since the regions where the decision makers highly disagree, the crisp decision (either acceptation or rejection) becomes less likely to be made. On the other hand, the formula is more "generous" in cases where the decision makers agree, even though without much confidence into the crisp decision. The considerations have been supported by an experimental illustration showing the possibility of the proposed fuzzy consensus aggregation operator to yield superior result when compared with classical aggregation operators.

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